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Two-dimensional fins attached to a thick wall—effect of non-uniform root temperature

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INTRODUCTION

FINS ARE the most convenient and easy way of increasing the heat transfer from a solid surface to a surrounding fluid. It is also a classical problem of the heat transfer literature which has been constantly revised over the years. The standard reference on fins is the book by Kern and Kraus [1].

The classical fin solution is based on a one-dimensional formulation which assumes that the temperature within the fin is uniform along its cross-section. This assumption is well justified when the convection coefficient between the fin and its surrounding fluid is small compared with the thermal conductivity of the fin material. The major parameter governing the accuracy of this classical solution is the transversal Biot number, $Bi = hW/k$, where W is half of the fin thickness, as shown in Fig. 1(a). Except for Bi much less than one, it has been shown [2–5] that this solution dos not yield accurate results, and that the errors rapidly increase with increasing Biot numbers. This limits the range of application of the one-dimensional approach.

An attempt to improve one-dimensional fin solutions was made by Aparecido and Cotta [6]. There the authors use the ideas of the coupled integral equation approach [7] to develop an approximate and accurate solution that takes into account the tempeature non-uniformity across the fin. The solution of ref. [6] retains the same degree of analytical involvement as for the classical approach but extends its range of applicability to considerably larger values of the Biot number.

Another point to be considered is that the temperature at the base of the fin is not really constant as is usually assumed. The presence of a fin attached to a solid surface causes temperature elevation or depression at the fin root depending upon whether the solid temperature is lower or higher than the temperature of the surrounding fluid, respectively [8, 9].

A first attempt to investigate the errors in the calculated magnitudes of heat lost from the fin due to non-uniformities of the base temperature was recently made by Look [10]. There the author performed a two-dimensional calculation and assumed an idealized non-constant fin root temperature of the form $T = T_0 + a\cos(\pi y/2W)$, where '$a$' was at most $\pm 0.2\ (T_0 - T_\infty)$. It was shown that non-uniformities on the root temperature strongly affect the fin performance.

Look also studied the effect of unequal top and bottom surface coefficients which are responsible for cross-sectional asymmetries in the temperature field along the fin. According to ref. [10], these asymmetries are not important when investigating the influence of root temperature variation on fin performance.

The work of Look provided insight into the effect of non-constant root temperature and motivated the present work. Here a more general situation is investigated. As is shown in Fig. 1(a), the fin is considered to be attached to a thick wall, as is the case in many engineering situations. A constant temperature is prescribed at the wall surface opposite to the fin. In this respect the fin root temperature is part of the solution and is not known a priori as is the case in ref. [10]. The problem is solved numerically using a finite volume methodology, and the results are compared with those obtained using the improved one-dimensional solution of ref. [6] as well as with the two-dimensional analytical solution, both for constant-root temperature.

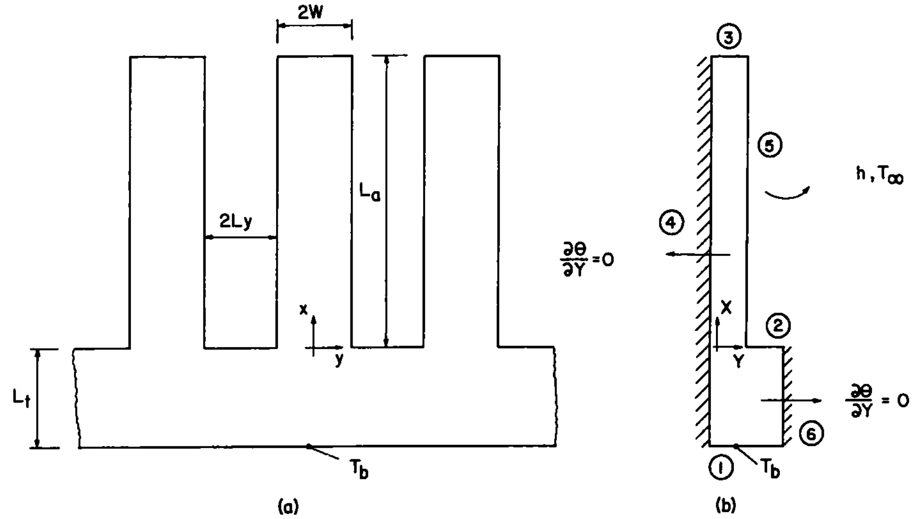

FIG. 1. Geometry and coordinate system for the problem. (a) Multifin array attached to a thick wall. (b) Solution domain.

NOMENCLATURE

- Bi Biot number, hW/k
- G_1 dimensionless fin length, L_a/W
- G_2 dimensionless wall thickness, L_t/W
- h convection coefficient
- k thermal conductivity
- L_a fin length
- L_t wall thickness
- L_y half of spacing between fins
- Q_0 dimensionless heat transfer rate at the base of the fin
- q_0 heat transfer rate at the base of the fin per unit length along the base
- T temperature
- T_b temperature at the wall surface opposite to where the fins are attached
- T_∞ ambient temperature
- W half of fin thickness
- X x/W
- x coordinate along fin length
- Y y/W
- y coordinate along fin cross-section.

Greek symbol
- Θ dimensionless temperature, $(T-T_\infty)/(T_b-T_\infty)$.

ANALYSES

The problem to be considered is shown in Fig. 1(a). As seen in the figure, a multifin array is attached to a wall whose inner surface is at T_b. The surrounding fluid is at T_∞, and the heat transfer coefficient between the fins and the fluid, h, is assumed to be constant throughout the fin. Because of the symmetry of the problem, the solution domain can be taken as being that of Fig. 1(b). All geometrical parameters are shown in Figs. 1(a) and (b).

For the rectangular two-dimensional fin shown in Fig. 1(b), the heat conduction equation for constant thermal conductivity can be written in dimensionless form as

$$\partial^2\Theta/\partial X^2+G_1^2\partial^2\Theta/\partial Y^2=0. \quad (1)$$

In writing equation (1), the following dimensionless variables and parameter were adopted:

$$\Theta=(T-T_\infty)/(T_b-T_\infty),\quad X=x/L_a,\quad Y=y/W,$$
$$G_1=L_a/W. \quad (2)$$

According to Fig. 1(b), equation (1) is to be solved subject to the following boundary conditions:

1. $X=-L_t/L_a,\quad 0\leqslant Y\leqslant 1+L_y/W\rightarrow\Theta=1$
2. $X=0,\quad 1\leqslant Y\leqslant 1+L_y/W\rightarrow\partial\Theta/\partial X+BiG_1\Theta=0$
3. $X=1,\quad 0\leqslant Y\leqslant 1\rightarrow\partial\Theta/\partial X+BiG_1\Theta=0$
4. $-L_t/L_a\leqslant X\leqslant 1,\quad Y=0\rightarrow\partial\Theta/\partial Y=0$
5. $0\leqslant X\leqslant 1,\quad Y=1\rightarrow\partial\Theta/\partial Y+Bi\Theta=0$
6. $-L_t/L_a\leqslant X\leqslant 0,\quad Y=1+L_y/W\quad\partial\Theta/\partial Y=0 \quad (3)$

where

$$Bi=hW/k. \quad (4)$$

In addition to Bi and G_1, two other dimensionless parameters govern the problem; they are the wall thickness G_2 and the fin spacing G_3:

$$G_2=L_t/W,\quad G_3=L_y/W. \quad (5)$$

For the present work, Bi was varied from 0.001 to 10, covering values that are commonly encountered in fin applications: G_1 was varied from 1 to 6 and G_2 ranged from 0.1 to 50. The dimensionless fin spacing G_3 was kept constant and equal to 6.

Once the temperature field has been calculated, the dimensionless heat transfer rate at the base of the fin, per unit length along the base, can be determined as

$$Q_0=q_0/[k(T_b-T_\infty)]=-\int_{-1}^{1}(\partial\Theta/\partial X)_{x=0}\,\mathrm{d}Y \quad (6)$$

where q_0 is the heat transfer rate at the base of the fin per unit length along the base.

The numerical solution of equation (1) was obtained using the finite volume methodology as described in ref. [11]. Use was made of cartesian coordinates and the domain irregularity was handled via the variable-conductivity method as suggested in ref. [12]. These are common practices in the heat transfer literature and need not be elaborated here.

The computational mesh varied depending on the Biot number investigated. For Bi values less than one, 30×30 grid points (X and Y directions) were employed, and for higher values of Bi the grid population increased, reaching a maximum of 60×60 points for $Bi=10$. The grids were uniformly spaced in the wall and in the fin.

RESULTS AND DISCUSSION

Presented in Table 1 are the results for the dimensionless heat transfer rates at the base of the fin, Q_0, for the limiting case where the wall thickness is zero ($G_2=0$) and a uniform temperature is imposed at the base of the fin. Values of Q_0 were calculated using the exact two-dimensional solution [13], the numerical model of the present work, the modified one-dimensional solution of ref. [6], and the classical one-dimensional solution. The errors associated to each one of the approximate solutions and the exact solution are also indicated in Table 1. As can be seen in the table, both the modified and the classical solution incur large errors at high Biot numbers.

For G_2 values other than zero, the temperature at the base of the fin will no longer be uniform. Results for the temperature profile along Y for $X=0$ are presented in Fig. 2. Each curve is for a given dimensionless wall thickness G_2.

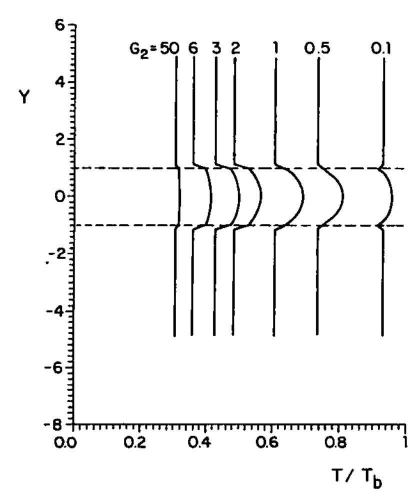

FIG. 2. Temperature profile along Y for $X=0$, having the dimensionless wall thickness, G_2, as a curve parameter; $Bi=1$, $G_1=6$ and $G_3=6$.

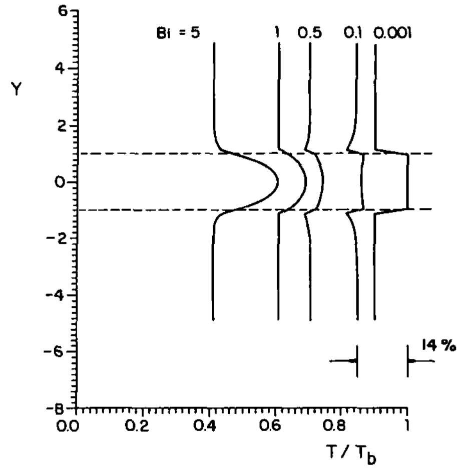

FIG. 3. Temperature profile along Y for $X = 0$, having the Biot number as curve parameter; $G_1 = 6, G_2 = 1$ and $G_3 = 6$.

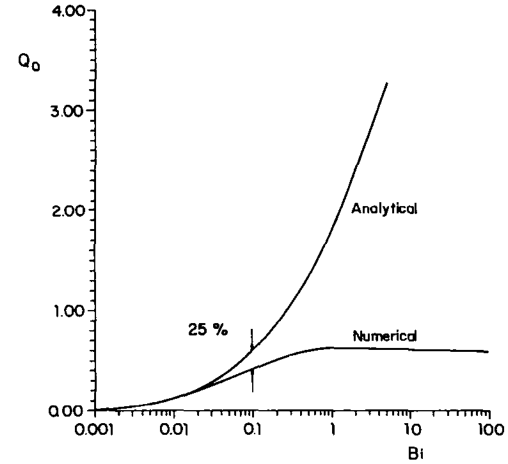

FIG. 4. Dimensionless heat transfer rate at the base of the fin as a function of the Biot number; $G_1 = 6, G_2 = 2$ and $G_3 = 6$.

The region for $-1 \leqslant Y \leqslant 1$, shown between the dashed lines, corresponds to the base of the fin. Except for very large or very small valus of G_2, the fin root temperature cannot be assumed as uniform.

The dependence on the Biot number of the temperature profile along Y for $X = 0$ is explored in Fig. 3. For all cases presented in the figure, $G_1 = 6$, $G_2 = 1$ and $G_3 = 6$. Again, the region corresponding to the base of the fin is shown between dashed lines. As expected, as the Biot number decreases the profile at the base of the fin gets flatter. However, the root fin temperature will only be that of the wall surface for Bi values less than 0.001. As seen in the figure, even for $Bi = 0.1$ the root fin temperature is approximately 14% smaller than T_b.

Further examination of each temperature profile in Fig. 3 shows that for high Bi values the fin acts as an insulator and the temperature at its base exhibits a parabolic shape, having its maximum value along the center of the fin. Also, the temperature along the base of the fin ($-1 \leqslant Y \leqslant 1$) is higher than the temperature at the wall ($Y > 1$ and $Y < -1$). As the Bi value decreases the temperature profile gets flatter and at $Bi = 0.001$ it is virtually flat. An interesting feature to be noted is that for $Bi = 0.1$ the temperature profile exhibits a curvature which is opposite to those for $Bi = 5$, 1 and 0.5, indicating the occurrence of the temperature depression effect [8, 9].

Results for the temperature profile along Y for $X = 0$ were also obtained for constant values of Bi, G_2 and G_3, and different values of the fin dimensionless length, G_1. It was observed that, for the range of values investigated here, the temperature profile at the base of the fin is little affected by G_1.

According to what was previously discussed, use of the wall temperature instead of the actual root temperature in calculating the heat transfer from the fin, even for Biot numbers as small as 0.1, can result in large errors. This is better explored with the aid of Fig. 4, which shows how the dimensionless heat transfer rate at the base of the fin, Q_0, varies with Bi.

The upper curve in Fig. 4 was obtained using the analytical two-dimensional solution, whereas the lower curve corrresponds to the numerical results of the present work. For the analytical solution the root fin temperature was taken as T_b. For both the analytical and the numerical results use was made of equation (6). It should be noted that this figure was prepared taking $G_2 = 2$; that is, wall thickness was equal to the fin thickness. As fins are usually very thin this represents a thin wall. In general, G_2 is greater than that, which would increase the deviations between the two curves. As seen from the figure, even for $Bi = 0.1$ the analytical two-dimensional solution overpredicts Q_0 by 25%. The deviations between the curves are due to temperature non-uniformities at the base of the fin as well as temperature depressions associated with the wall thermal resistance.

According to the results that have been shown, the inclusion of two-dimensional effects to account for temperature non-uniformities along the fin cross-section cannot be justified in practical situations if the root temperature is maintained uniform and equal to T_b. This is true even for fins attached to thin and well conducting walls. Exceptions occur for Bi values less than 0.01. However, for such small values of Bi good accuracy is achieved using the classical one-dimensional model, and there is no need to use more elaborate solutions, as observed in Table 1.

Another aspect to be explored from Fig. 4 is that the analytical solution Q_0 always increases with increasing Bi, whereas for the numerical solution Q_0 reaches a plateau beyond which it remains constant. This finding can be reasoned by noticing that for the analytical solution, because T_b is prescribed at the base of the fin, an increase in Bi can be seen as an increase in the heat transfer coefficient h, while k is kept constant. This necessarily leads to an increase in Q_0. For the numerical solution, the heat delivered through the fin to the surrounding fluid is bounded by the thermal resistance due to the wall, regardless of Bi.

For Biot numbers greater than 1, the heat transfer rate Q_0 tends to decrease for increasing values of Bi. This is so because the thermal resistance due to the fin causes the heat flow to deviate from the base of the fin, going directly from the heated wall at T_b to the cooled wall whose temperature approaches T_∞ as Bi increases.

CONCLUSIONS

The present work has dealt with two-dimensional fins attached to a thick wall. Effects of temperature non-uniformities at the base of the fin, as well as temperature depression due to the wall thermal resistance, are explored for various Biot numbers and different geometrical configurations.

The temperature profile at the base of the fin is highly dependent upon the Biot number and the dimensionless wall thickness, G_2. Except for very large or very small values of G_2, the fin root temperature cannot be assumed as uniform,

Table 1. Comparison between the dimensionless heat transfer rates at the base of the fin calculated using different models at uniform root temperature

		Exact†	Numerical‡		Modified§ [6]		Classical‖	
Bi	G_1	Q_0	Q_0	Error (%)	Q_0	Error (%)	Q_0	Error (%)
0.1	1	0.3547	0.3547	−0.0080	0.3547	−0.0080	0.3589	−1.2
0.1	5	0.5956	0.5955	−0.017	0.5970	0.23	0.6052	−1.6
1	1	1.852	1.850	0.073	1.823	−1.6	2.000	−8.0
1	5	1.806	1.799	−0.36	1.789	−0.93	2.000	−11
10	1	4.316	4.277	−0.90	3.548	−18	6.336	−47
10	5	4.091	3.952	−3.4	3.381	−17	6.325	−55

† Exact two-dimensional solution.
‡ Numerical solution of the present work.
§ Modified one-dimensional solution of ref. [6].
‖ Classical one-dimensional solution.

as is commonly adopted in the literature. For wall thickness equal to the fin thickness, it was shown that when the Bi values are less than 0.1 the root temperature is virtually uniform, but it can only be considered equal to the prescribed wall temperature for Bi less than 0.001.

It has been demonstrated that the common practice of including two-dimensional effects to account for temperature non-uniformities along the fin cross-section cannot be justified in practical situations if the root temperature is kept uniform. In situations where fin and wall have about the same thickness, only for Bi values less than 0.01 can the wall thermal resistance be neglected in calculating the heat transfer rate at the base of the fin. Usually, as walls are thicker than that, this limit has to be pushed further down. For such small values of Bi good accuracy can be achieved with the classical one-dimensional model and there is no need to employ more elaborate solutions.

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Natural convection heat transfer in smooth and ribbed vertical channels

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INTRODUCTION

NATURAL convection heat transfer in smooth and ribbed vertical channels is of interest in electronic cooling applications. The primary purpose of the present paper is to study natural convection heat transfer in both smooth and ribbed vertical channels, and to determine the effect of two-dimensional ribs on the smooth channel heat transfer.

As compared to the abundant literature on smooth channels (see reviews by Jaluria [1], Aihara [2] and Moffat and Ortega [3]), relatively few studies exist on natural convection in ribbed channels. Moffat and Ortega [3] and Incropera [4] have reviewed the pertinent literature. Ortega and Moffat [5, 6] have studied free convection heat transfer past a vertical adiabatic surface with three-dimensional heated protrusions. Shakerin *et al.* [7] numerically studied the effect